SUPER-REFLEXIVITY AND THE GIRTH OF SPHERES

BY

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ABSTRACT

It is shown that a Banach space is super-reflexive if and only if the girth of its unit ball is greater than 4. Consequently, "girth greater than 4" is a property preserved under isomorphisms and duality.

1. Introduction

This note is a further contribution to the geometrical insight into certain conditions, stronger than reflexivity, for Banach spaces. The concept of *super-reflexivity* of Banach spaces was introduced and discussed by James [4], who showed that it has several equivalent geometric interpretations and that it is preserved under isomorphisms and duality. The concept of the *girth* of the unit ball, i.e., the infimum of the lengths of centrally symmetric simple closed rectifiable curves on its surface, was introduced by Schäffer [6]; Schäffer and Sundaresan proved in [8] that a Banach space is reflexive if the girth of its unit ball is not 4. In this note we prove that, in fact, the space is super-reflexive *if and only if* this girth is not 4. We thus obtain, on the one hand, an additional metric characterization of super-reflexivity, and prove, on the other, that "girth not 4" is a property preserved under isomorphism and duality.

2. Definitions and auxiliary results

All normed spaces shall be non-trivial real normed linear spaces. A *subspace* of a normed space is a linear subspace provided with the norm induced by the inclusion.

If X and Y are normed spaces, Y is said to be *finitely representable in* X if

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and only if for every finite-dimensional subspace Z of Y and every number $\lambda > 1$ there exists a linear mapping $T: Z \to X$ such that $\lambda^{-1}\| z \| \leqq \| Tz \| \leqq \lambda \| z \|$ for all $z \in Z$; equivalently, if and only if for every finite-dimensional subspace Z of X and every number $\varepsilon > 0$ there exists a subspace W of X, of the same dimension, such that $\Delta(Z, W) \leqq \varepsilon$, where Δ is the Banach-Mazur distance [1, pp. 242–243]. Finite representability is a transitive relation, and it is easy to see that a normed space is finitely representable in every dense subspace of itself. It follows that Y is finitely representable in X if and only if the completion of Y is.

A Banach space X is said to be *super-reflexive* [4] if and only if no non-reflexive Banach space is finitely representable in X. The preceding comments show that to prove that a given Banach space X is *not* super-reflexive it is enough to exhibit a normed space that has a non-reflexive completion and is finitely representable in X.

For a normed space X, we denote by $2m(X)$ the *girth* of its unit ball, as defined in the introduction. For a more detailed discussion of this concept, see [6], [8]. Obviously, $m(X) \geqq 2$. The link between the condition "$m(X) = 2$" and the geometric conditions that prevent the space from being super-reflexive is the following property of a normed space X, introduced in [8]; it is a generalized negation of uniform non-squareness [2]:

(J): *For every positive integer n and every number ρ, $0 < \rho < 1$, there exist $x_k \in X$, $k = 1, \cdots, n$, such that $\| x_k \| \leqq 1$ for $k = 1, \cdots, n$, and*

$$\left\| - \sum_{1}^{j} x_k + \sum_{j+1}^{n} x_k \right\| > \rho n$$

for $j = 0, \cdots, n$.

We require several auxiliary results; the first two are theorems from [8].

THEOREM 2.1. ([8, Theorem 2.2]). *If Y is a non-reflexive Banach space, then Y satisfies* (J).

THEOREM 2.2. ([8, Theorem 3.2]). *A normed space X satisfies* (J) *if and only if $m(X) = 2$.*

LEMMA 2.3. *Let X be a normed space, ε a number, $0 < \varepsilon < 1$, and n a positive integer. If $u_k \in X$ and $\| u_k \| \leqq 1$ for $k = 1, \cdots, n$, and if $\| \Sigma_1^n u_k \| > n - \varepsilon$, then $\| \Sigma_1^n \gamma_k u_k \| > (1 - \varepsilon)\, \Sigma_1^n \gamma_k$, for every choice of numbers $\gamma_k \geqq 0$, $k = 1, \cdots, n$, other than $\gamma_1 = \cdots = \gamma_n = 0$.*

PROOF. Let the γ_k be given, and set $\gamma_{n+k} = \gamma_k$, $k = 1,\cdots,n-1$. Then

$$(n-\varepsilon)\sum_1^n \gamma_k < \left\| \sum_1^n u_k \right\| \sum_1^n \gamma_k = \left\| \sum_{i=0}^{n-1} \sum_{k=1}^n \gamma_{k+i} u_k \right\| \leqq \left\| \sum_1^n \gamma_k u_k \right\|$$

$$+ \sum_{i=1}^{n-1} \sum_{k=1}^n \gamma_{k+i} \| u_k \| \leqq \left\| \sum_1^n \gamma_k u_k \right\| + (n-1) \sum_1^n \gamma_k,$$

and the conclusion follows.

LEMMA 2.4. *Let E be a real linear space and let $\pi: E \to R$ be a seminorm on E. If n is a positive integer and $x_k \in E$, $k = 1,\cdots,n$, satisfy*

$$\pi(x_k) \leqq 1, \qquad k = 1,\cdots,n, \tag{2.1}$$

$$\pi\left(-\sum_1^j x_k + \sum_{j+1}^n x_k \right) > n-1, \qquad j = 0,\cdots,n, \tag{2.2}$$

then $\pi(\sum_1^n \alpha_k x_k) > 0$ for all real α_k, $k = 1,\cdots,n$, unless $\alpha_1 = \cdots = \alpha_n = 0$.

PROOF. (cf. [7, Proof of Theorem 2]). Suppose that, on the contrary, there exist α_k, not all 0, such that

$$\pi\left(\sum_1^n \alpha_k x_k \right) = 0. \tag{2.3}$$

We may assume without loss of generality that

$$\max_k |\alpha_k| = 1 = |\alpha_h| \tag{2.4}$$

for some h, $1 \leqq h \leqq n$. Then

$$\left| -\sum_1^{h-1} \alpha_k + \sum_h^n \alpha_k \right| + \left| -\sum_1^h \alpha_k + \sum_{h+1}^n \alpha_k \right| \geqq \left| \left(-\sum_1^{h-1} \alpha_k + \sum_h^n \alpha_k \right) \right.$$

$$\left. - \left(-\sum_1^h \alpha_k + \sum_{h+1}^n \alpha_k \right) \right| = 2|\alpha_h| = 2.$$

One of the two summands of the leftmost member is not less than 1; setting $j = h-1$ or $j = h$, and replacing *every* α_k by $-\alpha_k$ if necessary, we may consequently assume, without invalidating (2.3), (2.4), that

$$-\sum_1^j \alpha_k + \sum_{j+1}^n \alpha_k \geqq 1 \text{ for some } j,\ 0 \leqq j \leqq n. \tag{2.5}$$

Combining (2.2) for that value of j with (2.3), (2.4), (2.1), (2.5), we find

$$n-1 < \pi\left(-\sum_1^j x_k + \sum_{j+1}^n x_k \right) = \pi\left(-\sum_1^j (1+\alpha_k) x_k + \sum_{j+1}^n (1-\alpha_k) x_k \right)$$

$$\leqq \sum_{1}^{j} (1+\alpha_k) + \sum_{j+1}^{n} (1-\alpha_k) = n - \left(- \sum_{1}^{j} \alpha_k + \sum_{j+1}^{n} \alpha_k \right) \leqq n-1,$$

a contradiction.

3. The main theorem

THEOREM 3.1. *If X is a Banach space, X is super-reflexive if and only if X does not satisfy* (J).

PROOF. 1. Suppose that X is not super-reflexive. Then there exists a non-reflexive Banach space Y that is finitely representable in X; by Theorem 2.1, Y satisfies (J). Let n and ρ, $0 < \rho < 1$, be given, and choose $\lambda > 1$ such that $\lambda^2\rho < 1$. There exist $y_k \in Y$, $k = 1,\cdots,n$, such that $\| y_k \| \leqq 1$ for $k = 1,\cdots,n$ and $\| -\Sigma_1^j y_k + \Sigma_{j+1}^n y_k \| \geqq \lambda^2 \rho n$ for $j = 0,\cdots,n$. If Z is the (at most n-dimensional) subspace of Y spanned by $y_1,\cdots,y_n$, there exists a linear mapping $T: Z \to X$ such that $\lambda^{-1}\| z \| \leqq \| Tz \| \leqq \lambda \| z \|$ for all $z \in Z$. We set $x_k = \lambda^{-1}Ty_k \in X$, $k = 1,\cdots,n$, and find $\| x_k \| \leqq \lambda^{-1}\lambda \| y_k \| \leqq 1$ for $k = 1,\cdots,n$, and

$$\left\| - \sum_{1}^{j} x_k + \sum_{j+1}^{n} x_k \right\| \geqq \lambda^{-2} \left\| - \sum_{1}^{j} y_k + \sum_{j+1}^{n} y_k \right\| > \rho n$$

for $j = 0,\cdots,n$. Since n and ρ, $0 < \rho < 1$, were arbitrary, X satisfies (J).

2. In the rest of the proof we assume that X satisfies (J). There exist, therefore, $x_k^n \in X$ for all $k = 1,\cdots,n$ and $n \geqq 1$ such that

$$\| x_k^n \| \leqq 1 \text{ for all } k = 1,\cdots,n \text{ and } n \geqq 1 \tag{3.1}$$

and $\| -\Sigma_{k=1}^j x_k^n + \Sigma_{k=j+1}^n x_k^n \| > n - n^{-1}$ for all $j = 0,\cdots,n$ and $n \geqq 1$. From Lemma 2.3 we conclude that

$$\left\| - \sum_{k=1}^{j} \gamma_k x_k + \sum_{k=j+1}^{n} \gamma_k x_k^n \right\| > (1-n^{-1}) \sum_{k=1}^{n} \gamma_k \quad \text{for all } j = 0,\cdots,n; \text{ all}$$

(3.2)

choices of $\gamma_k \geqq 0$ other than $\gamma_1 = \cdots = \gamma_n = 0$; and all $n \geqq 1$.

3. Let E be a linear space with a countable Hamel basis $\{e_k: k = 1,2,\cdots\}$, and let E_m be the m-dimensional linear subspace spanned by $e_1,\cdots,e_m$, for each $m = 1,2,\cdots$.

For each m, consider the functions $f_{mn}: E_m \to R$, $n \geqq m$, defined by

$$f_{mn}\left(\sum_{k=1}^{m} \alpha_k e_k \right) = \left\| \sum_{k=1}^{m} \alpha_k x_k^n \right\|. \tag{3.3}$$

Each one of these functions is a seminorm (actually a norm, but we do not need this fact). For fixed m, the sequence $(f_{mn})_{n \geqq m}$ is uniformly equicontinuous, and uniformly bounded on compact sets (all these terms refer to the natural separated uniformity of the m-dimensional space E_m), since

$$\left| f_{mn}\left(\sum_{k=1}^{m} \alpha_k e_k\right) - f_{mn}\left(\sum_{k=1}^{m} \beta_k e_k\right)\right| \leqq \sum_{k=1}^{m} |\alpha_k - \beta_k|$$

on account of (3.3) and (3.1). Therefore an application of Ascoli's Theorem for each m, combined with a diagonal process, yields the existence of a strictly increasing sequence $(p(n))$ of positive integers such that the limit

$$\pi\left(\sum_{k=1}^{m} \alpha_k e_k\right) = \lim_{n\to\infty} f_{m,p(n)}\left(\sum_{k=1}^{m} \alpha_k e_k\right) = \lim_{n\to\infty} \left\| \sum_{k=1}^{m} \alpha_k x_k^{p(n)} \right\| \tag{3.4}$$

exists for all $\sum_{k=1}^{m} \alpha_k e_k \in E$. (We could have obtained this result by a single appeal to Ascoli's Theorem for functions on the space E provided with the direct sum uniformity.) From (3.4) it is clear that $\pi: E \to R$ is a seminorm. We shall now show that it actually is a norm.

4. From (3.1), (3.2), (3.4) we deduce

$$\pi(e_k) \leqq 1 \qquad \text{for all } k \geqq 1, \tag{3.5}$$

$$\pi\left(-\sum_{k=1}^{j} \gamma_k e_k + \sum_{k=j+1}^{m} \gamma_k e_k\right) \geqq \sum_{k=1}^{m} \gamma_k \qquad \text{for all } j = 0, \cdots, m; \tag{3.6}$$

all choices of $\gamma_k \geqq 0$, $k = 1, \cdots, m$; and all $m \geqq 1$.

In particular,

$$\pi\left(-\sum_{k=1}^{j} e_k + \sum_{k=j+1}^{m} e_k\right) \geqq m \text{ for all } j = 0, \cdots, m \text{ and all } m \geqq 1. \tag{3.7}$$

It follows from (3.7) and Lemma 2.4 that π is a *norm* on E, as claimed. From now on, E shall be taken to be the given linear space provided with the norm π. We remark in passing that the triangle inequality compels equality in (3.5), (3.6), (3.7).

5. We claim that E is finitely representable in X. Since every finite-dimensional subspace of E is a subspace of E_m for a suitable m, it is enough to show that, for each m and each number $\lambda > 1$, there exists a linear mapping $T: E_m \to X$ such that $\lambda^{-1}\pi(z) \leqq \| Tz \| \leqq \lambda\pi(z)$ for all $z \in E_m$.

In the argument leading to (3.4), the use of Ascoli's Theorem allows us to assert that the convergence of the sequence $(f_{m,p(n)})_n$ to the restriction of π to E_m is *uniform* on compact subsets of E_m. Since π is a norm, the set $\{z \in E_m: \pi(z) = 1\}$ is compact; since the $f_{m,p(n)}$ and π, being seminorms, are absolutely homogeneous, it follows that there exists a positive integer n^* such that

$$\lambda^{-1}\pi(z) \leqq f_{m,p(n^*)}(z) \leqq \lambda\pi(z) \text{ for all } z \in E_m. \tag{3.8}$$

We then define $T: E_m \to X$ by $T(\sum_{k=1}^{m} \alpha_k e_k) = \sum_{k=1}^{m} \alpha_k x_k^{p(n^*)}$. Thus T is linear and, by (3.3), $\| Tz \| = f_{m,p(n^*)}(z)$ for all $z \in E_m$. In view of (3.8), T satisfies all the required conditions.

6. To conclude, we claim that the completion of E is not reflexive. Indeed, its unit ball contains the vectors $e_1, e_2, \cdots$, and (3.6) implies

$$\operatorname{dist}(\operatorname{conv}\{e_1, \cdots, e_j\}, \operatorname{conv}\{e_{j+1}, e_{j+2}, \cdots\}) = 2 \text{ for all } j \geqq 1.$$

Therefore this space is not reflexive [3, Theorem 8: equivalence of (29) and (32)].

The normed space E is finitely representable in X and its completion is not reflexive. Therefore X is not super-reflexive, and the proof is complete.

Added in proof. A comment by L. A. Karlovitz led to the realization that the proof of the "only if" part of this theorem could have been concluded after formula (3.2) with the observation that

$$\operatorname{dist}(\operatorname{conv}\{x_1, \cdots, x_j\}, \operatorname{conv}\{x_{j+1}, \cdots, x_n\}) > 2(1 - n^{-1}), \ j = 1, \cdots, n-1,$$

so that X has a "finite flatness property" and is therefore not super-reflexive [5; Lemmas B and C]. The remainder of our proof is then an instructive alternative proof of this last implication.

COROLLARY 3.2. *A normed space has a super-reflexive completion if and only if* $m(X) > 2$.

PROOF. It is obvious that a normed space satisfies (J) if and only if its completion satisfies (J). The conclusion follows from Theorems 3.1 and 2.2.

COROLLARY 3.3. *If X and Y are isomorphic normed spaces, then $m(X) > 2$ if and only if $m(Y) > 2$. If X is a normed space, $m(X) > 2$ if and only if $m(X^*) > 2$.*

PROOF. Super-reflexivity of Banach spaces is preserved under isomorphism and duality (and implies reflexivity) [4, Theorem 2]. The conclusion follows from Corollary 3.2.

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